Experimental Design

Basic Goal:
- Find out which factor contributes how much
- Do this before and during the analysis and not just after the analysis.

Advanced Goal:
- if you have to reduce experiments do this in a clever way.

Example of a $2^7-4$ design

<table>
<thead>
<tr>
<th>Experiment No.</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Properties:
1. The sum of each column is zero:
   $$\sum_i x_{ij} = 0 \quad \forall j$$
   where $x_{ij}$ represents the level of the $j$th variable in the $i$th experiment.
2. The sum of the products of any two columns is zero:
   $$\sum_i x_{ij} x_{il} = 0 \quad \forall j \neq l$$
3. The sum of the squares of each column is $2^7-4=8$, that is, &
   $$\sum_i x_{ij}^2 = 8 \quad \forall j$$

Computing Effects

- Estimate $y$
- Effective responses $y_i$

Using the orthogonality property of the factor levels chosen, it can be shown that

$$q_A = \frac{\sum_i y_i x_{i1}}{8} = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8$$

Similarly,

$$q_B = \frac{\sum_i y_i x_{i2}}{8} = y_1 - y_2 + y_3 - y_4 - y_5 + y_6 - y_7 + y_8$$

and so on.

• Orthogonality
Example:

- The effect of factors A-G on variation:
  37.26, 4.74, 43.40, 6.75, 0, 8.06, 0.03
- A and C explain almost everything

No interactions just Factors
- proceed if interactions are negligible
- saves lot of work

Preparing the table

1. Choose \( k - p \) factors and prepare a complete sign table for a full factorial design with \( k - p \) factors. This will result in a table of \( 2^{k-p} \) rows and \( 2^{k-p} \) columns. The first column will be marked \( I \) and consists of all 1's. The next \( k - p \) columns will be marked with the \( k - p \) factors that were chosen. The remaining columns are simply products of these factors.

2. Of the \( 2^{k-p} - k + p - 1 \) columns on the right, choose \( p \) columns and mark them with the \( p \) factors that were not chosen in step 1.

Example 2

4-1 design

Cofounding

\[ D = ABC \text{ but also } A = BCD \]

Complete list:

\[ A = BCD, \quad B = ACD, \quad C = AB, \quad AB = CD \]

Alternate design:

List of cofoundings:

\[ I = ABD, \quad A = BD, \quad B = AD, \quad C = ABCD \]

\[ D = AB, \quad AC = BCD, \quad BC = ACD, \quad ABC = CD \]
Algebra of Cofounding

- Ring or Field structures with generators similar to finite fields over GF[2].

Rules

1. The mean \( I \) is treated as unity. For example, \( I \) multiplied by \( A \) is \( A \).
2. Any term with a power of 2 is erased. For example, \( AB^2C \) is the same as \( AC \).

Let us illustrate this with the first design, which has \( I = ABCD \).

Multiplying both sides by \( A \), we get
\[
A = A^2BCD = BCD
\]

Multiplying both sides by \( B, C, D, \) and \( AB \), we get
\[
B = AB^2CD = ACD
\]
\[
C = ABC^2D = ABD
\]
\[
D = ABCD^2 = ABC
\]
\[
AB = A^2B^2CD = CD
\]

and so on.

The polynomial \( I = ABCD \) used to generate all confoundings for this design is called the generator polynomial for this design. Similarly, the generator polynomial for the second design is \( I = ABC \).

Example of a Generator:

\[
D = AB, \quad E = AC, \quad F = BC, \quad G = ABC
\]

Multiplying each of the four equations by their left-hand sides, we get
\[
I = ABD, \quad I = ACE, \quad I = BCF, \quad I = ABCG
\]

Or equivalently,
\[
I = ABD = ACE = BCF = ABCG
\]

The product of any subset of the preceding terms is also equal to \( I \). Thus the complete generator polynomial is
\[
I = ABD = ACE = BCF = ABCG = BCDEF = CDGF
\]
\[
= CDG = ABEF = BEG = AFG = DEF = DFG = ABDF = BDG = ACEFG = BCDEFG
\]

Other confoundings for the design can be obtained by multiplying this equation by \( A, B, \ldots \). For example,
\[
A = BD = CE = ABCF = BCG = ABCDE = CDF
\]
\[
= ACDG = BDEF = ABEFG = DFG = ADEF = DFG = ABDFG = BDG = ACEFG = BCDEFG
\]

Design resolution:

- Degree of that generator Polynomial
- Min over all cofounding expressions: Order of cofounded effects left plus sum of order cofounded effects right.

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Latex vs. troff

Case Study 19.1 The CPU time taken by two text-formatting programs, called \LaTeX{} and troff, was measured using synthetic files of various sizes and complexity levels. Six factors, each with two levels were chosen for the study. The first two factors were the text-formatting programs and size of the files. The remaining four factors were number of equations, floats, tables, and footnotes in the file. The assignment of factors and their levels is shown in Table 19.6. A \( 2^{k-1} \) factorial fractional design with the generator polynomial \( I = BCDEF \) was used. The largest effects and interactions, computed by the sign table method, are shown in Table 19.7. The following conclusions can be reached from these results:

1. Over 90% of the variation is explained by the three factors Bytes, Program, and Equations and a second-order interaction.
2. The text file sizes (in bytes) in these experiments were significantly different, making the effect more than that of the text-formatting programs being compared.
3. The high percentage of variation explained by the “program \times Equation” interaction indicates that the choice of the text-formatting program depends upon the number of equations in the text. If we consider only the programs and equations in isolation, the relative amount of CPU time for various combinations is shown in Table 19.8. This shows that troff takes too much CPU time if there are equations in the text.
4. The “Program \times Bytes” interaction is low. This indicates that changing the file size affects both programs in a similar manner.
5. If possible, the experiments should be redone with a reduced range of file sizes so that the programs rather than the workload come out as the most significant factor. Alternately, the number of levels of file sizes should be increased.
One Factor Experiments

• More than two levels

Statistical model

• r observations [i]

• a alternatives [j]

• ra measurements [i,j]

• α_j effects

• e_{ij} errors

The model used in single-factor designs is

\[ y_{ij} = \mu + \alpha_j + e_{ij} \] (20.1)

Here, \( y_{ij} \) is the \( i \)th response (or observation) with the factor at level \( j \) (that is, the \( j \)th alternative), \( \mu \) is the mean response, \( \alpha_j \) is the effect of alternative \( j \), and \( e_{ij} \) is the error term. The effects are computed so that they add up to zero:

\[ \sum \alpha_j = 0 \]

Computation of the effects:

If we substitute the observed responses in the model Equation we obtain \( ar \) equations. Adding these equations, we get:

\[ \sum_{i=1}^{r} \sum_{j=1}^{a} y_{ij} = ar\mu \]

Since the effects \( \alpha_j \) add up to zero (by design) and we want the mean error to be zero, the preceding equation becomes:

\[ \sum_{i=1}^{r} \sum_{j=1}^{a} y_{ij} = ar\mu + 0 + 0 \]

The model parameter \( \mu \) is therefore given by:

\[ \mu = \frac{1}{ar} \sum_{i=1}^{r} \sum_{j=1}^{a} y_{ij} \]

The quantity on the right-hand side is the so-called grand mean of all \( ar \) responses. It is denoted by \( \overline{y} \).

\[ \overline{y}_j = \frac{1}{r} \sum_{i=1}^{r} y_{ij} \]

Substituting \( \mu + \alpha_j + e_{ij} \) for \( y_{ij} \), we obtain:

\[ \overline{y}_j = \frac{1}{r} \sum_{i=1}^{r} (\mu + \alpha_j + e_{ij}) \]

\[ = \frac{1}{r} \left( r\mu + ra\alpha + r\sum_{i=1}^{a} e_{ij} \right) \]

\[ = \mu + \alpha_j \]

Here we have assumed that the error terms for \( r \) observations belonging to each alternative add up to zero. The parameter \( \alpha_j \) can thus be estimated as follows:

\[ \alpha_j = \overline{y}_j - \mu = \overline{y}_j - \overline{y} \]

Example:

Example 20.1 In a code size comparison study, the number of bytes required to code a workload on three different processors R, V, and Z was measured five times each (each time a different programmer was asked to code the same workload).

<p>| TABLE 20.1 Data from a Code Size Comparison Study |</p>
<table>
<thead>
<tr>
<th>R</th>
<th>V</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>144</td>
<td>101</td>
<td>130</td>
</tr>
<tr>
<td>120</td>
<td>144</td>
<td>180</td>
</tr>
<tr>
<td>176</td>
<td>211</td>
<td>141</td>
</tr>
<tr>
<td>288</td>
<td>288</td>
<td>374</td>
</tr>
<tr>
<td>144</td>
<td>72</td>
<td>302</td>
</tr>
</tbody>
</table>

Results:

• Average 187.7 bytes


Errors

Once the model parameters have been computed, we can estimate the response for each of \( a \) alternatives. The estimated response for the \( j \)th alternative is given by

\[ \hat{y}_j = \mu + \alpha_j \]

The difference between the measured and the estimated response represents experimental error. If we compute experimental errors in each of the \( ar \) observations, the mean error should come out zero since the parameter values \( \mu \) and \( \alpha_j \) were computed assuming the sum of errors for each column was zero. The variance of the errors can be estimated from the Sum of Squared Errors (SSE):

\[ SSE = \sum_{i=1}^{r} \sum_{j=1}^{a} e^2_{ij} \]

Example 20.2 Computation of errors for the code size comparison study of Example 20.1 is as follows:

<table>
<thead>
<tr>
<th>[144 101 130]</th>
<th>[187.7 187.7 187.7]</th>
<th>[-13.3 -24.5 37.7]</th>
</tr>
</thead>
<tbody>
<tr>
<td>120 144 180</td>
<td>187.7 187.7 187.7</td>
<td>-13.3 -24.5 37.7</td>
</tr>
<tr>
<td>176 211 141</td>
<td>187.7 187.7 187.7</td>
<td>-13.3 -24.5 37.7</td>
</tr>
<tr>
<td>288 288 374</td>
<td>187.7 187.7 187.7</td>
<td>-13.3 -24.5 37.7</td>
</tr>
<tr>
<td>144 72 302</td>
<td>187.7 187.7 187.7</td>
<td>-13.3 -24.5 37.7</td>
</tr>
<tr>
<td>Column sum [ \sum y_j = 872 ]</td>
<td>Column sum [ \sum y_j = 816 ]</td>
<td>Column sum [ \sum y_j = 1127 ]</td>
</tr>
<tr>
<td>Column mean [ \overline{y}_j = 174.4 ]</td>
<td>Column mean [ \overline{y}_j = 163.2 ]</td>
<td>Column mean [ \overline{y}_j = 225.4 ]</td>
</tr>
<tr>
<td>Column effect [ \alpha_1 = \overline{y}_j - \overline{y} ]</td>
<td>Column effect [ \alpha_2 = \overline{y}_j - \overline{y} ]</td>
<td>Column effect [ \alpha_3 = \overline{y}_j - \overline{y} ]</td>
</tr>
<tr>
<td>= -13.3</td>
<td>= -24.5</td>
<td>= 37.7</td>
</tr>
</tbody>
</table>

Each observation has been broken into three parts: a grand mean \( \mu \), the processor effect \( \alpha_j \)'s, and the residuals. A matrix notation is used for all three parts. The sum of squares of entries in the residual matrix is

\[ SSE = (-30.4)^2 + (-54.4)^2 + \ldots + (76.6)^2 = 94,365.20 \]
Allocation of Variation

\[ y_{ij} = \mu + \alpha_i + \epsilon_{ij} + 2 \mu \alpha_i + 2 \mu \epsilon_{ij} + 2 \alpha_i \epsilon_{ij} \]

Adding corresponding terms of ar such equations, we obtain

\[ \sum_{i,j} y_{ij}^2 = \sum_{i,j} \mu^2 + \sum_{i,j} \alpha_i^2 + \sum_{i,j} \epsilon_{ij}^2 + \text{cross-product terms} \]

The cross-product terms all add to zero due to the constraints that the effects add to zero (\( \sum \alpha_i = 0 \)) and that the errors for each column add to zero (\( \sum \epsilon_{ij} = 0 \)). The preceding equation, expressed in terms of sums of squares, can be written as

\[ SSY = SS0 + SSA + SSE \]

where SSY is the sum of squares of \( y \), SS0 is the sum of squares of grand means, SSA is the sum of squares of effects, and SSE is the sum of square errors. Note that SS0 and SSA can be easily computed as follows:

\[ SS0 = \sum_{i=1}^{r} \sum_{j=1}^{s} \mu_i^2 = ar \mu^2 \]

\[ SSA = r \sum_{i=1}^{r} \sum_{j=1}^{s} \alpha_i^2 = r \sum_{j=1}^{s} \alpha_j^2 \]

Thus, SSE can be calculated easily from SSY without calculating individual errors.

The total variation of \( y \) (SST) is defined as

\[ SST = \sum_{i,j} (y_{ij} - \bar{y})^2 = \sum_{i,j} y_{ij}^2 - ar \bar{y}^2 = SSY - SS0 = SSA + SSE \]

The total variation can therefore be divided into two parts, SSA and SSE.

Analysis of Variance (ANOVA)

- Significance Test
- Chi-square Test
- Incorporate distribution and degrees of freedom.

Confidence Intervals

Confidence Intervals for

\[ \alpha_j \] is something above/below average with 95% confidence.

\[ \alpha_i - \alpha_j \] is a difference relevant with 95% confidence.

Do calculations according to receipe in book.
1. Model: \( y_{ij} = \mu + \alpha_i + \epsilon_{ij} \); the effects are computed so that
\[ \sum_{i=1}^{a} \alpha_i = 0. \]

2. Effects: \( \mu = \bar{y} = \frac{\sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij}}{\sum_{i=1}^{a} n_i} \),
\( \alpha_i = y_{i.} - \bar{y} , \quad j = 1, 2, \ldots, a \)

3. Allocation of variation: SSE can be calculated after computing other terms below:
\[ \begin{align*}
\sum_{ij} y_{ij}^2 & = a \sigma^2 + \tau \sum_{i} \alpha_i^2 + \sum_{ik} \epsilon_{ik}^2 \\
SSE & = SSI + SSA + SSE
\end{align*} \]

4. Degrees of freedom: \( SSI = S0 + SSA + SSE \)
\( \tau = 1 + \frac{(a-1) + a(r-1)}{ar} \)

5. Mean squares: \( MSA = SSA/(a-1) ; MSE = SSE/(ra(r-1)) \)

6. Analysis of variance: MSA/MSE should be greater than \( F_{\alpha/2, a-1, ra(r-1)} \)

7. Standard deviation of errors: \( \sigma = \sqrt{\text{MSE}} \)

8. Standard deviation of parameters: \( s_{\alpha}^2 = \tau^2 / \sigma \), \( s_{\epsilon}^2 = \tau^2 (a - 1) / ar \)

9. Contrast of effects \( \sum_{j=1}^{a} n_j y_{ij} \), where \( \sum_{i} n_i \alpha_i = 0 \).
\[ \begin{align*}
\text{Mean} & = \sum_{i=1}^{a} \frac{n_i y_{i.}}{\sum_{i} n_i} \\
\text{Variance} & = \sum_{i=1}^{a} \frac{n_i y_{i.}^2}{ar}
\end{align*} \]

10. All confidence intervals are computed using \( t_{\alpha/2; a(r-1)} \).

11. Model assumptions:
   (a) Errors are IID normal variates with zero mean.
   (b) Errors have the same variance for all factor levels.
   (c) The effect of the factor and errors are additive.

12. Visual tests:
   (a) The scatter plot of errors versus predicted responses should not have any trend.
   (b) The normal quantile-quantile plot of errors should be linear.
   (c) Spread of \( y \) values for all levels of the factor should be comparable.

If any test fails or if the ratio \( y_{\text{max}}/y_{\text{min}} \) is large, multiplicative models or transformations should be investigated.